

Uniform hyperbolicity in nonflat billiards

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Abstract

Uniform hyperbolicity is a strong chaotic property which holds, in particular, for Sinai billiards. In this paper, we consider the case of a nonflat billiard, that is, a Riemannian manifold with boundary. Each trajectory follows the geodesic flow in the interior of the billiard, and bounces when it meets the boundary. We give a sufficient condition for a nonflat billiard to be uniformly hyperbolic. As a particular case, we obtain a new criterion to show that a closed surface has an Anosov geodesic flow.

1 Notations

In this paper, a *smooth billiard* is a compact subset D of a Riemannian surface M , such that D has a smooth boundary while M has no boundary. Hence, each component of ∂D is the image of a smooth embedding $\Gamma : \mathbb{T}^1 \rightarrow M$ (with unit speed). Each curve Γ is called a *wall* of D : it has a unit tangent vector T and a unit normal vector N pointing toward $\text{Int } D$. The curvature of Γ is $\left\langle \frac{dT}{dt} \mid N \right\rangle$. A billiard whose walls have negative curvature is said to be *dispersing*.

Most of the time, the authors study *flat billiards*, and more precisely, billiards in the ambient manifold $M = \mathbb{R}^2$ or $M = \mathbb{T}^2$, but here, our aim is to understand how chaos may appear in billiards which are not flat.

One defines the phase space $\Omega = T^1(\text{Int } D)$, and the billiard flow $\phi_t : \Omega \rightarrow \Omega$, in the following way:

1. As long as it does not hit a wall, the particle follows a geodesic in M ;
2. When it arrives to the boundary of the billiard, the particle bounces, following the billiard reflection law: the angle between the particle's speed vector and the boundary's tangent line is preserved (Figure 1).

The flow ϕ_t is not defined at all times :

1. It is not defined at times when the particle is on the boundary of the billiard. Of course, one could extend the definition to such t , but the flow obtained in this way would not be continuous¹.

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¹Many authors change the topology of Ω in order to make the flow continuous, but it cannot be made differentiable.

2. When the particle makes a grazing collision with a wall at a time $t_0 > 0$, *i.e.* collides with the boundary with an angle $\theta = 0$, the flow stops being defined for all times $t \geq t_0$. Although one could extend continuously the definition of the trajectory after such a collision, the differentiability of the flow would be lost.

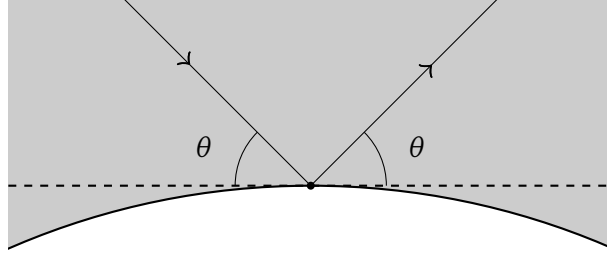


Figure 1: The billiard reflection law.

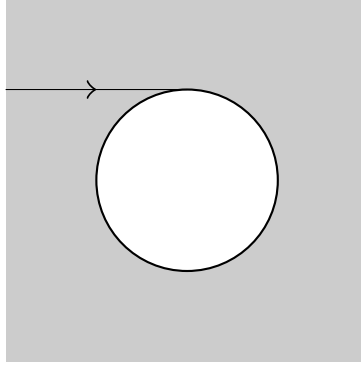


Figure 2: A grazing collision on a dispersing billiard in \mathbb{T}^2 . The flow stops being defined after this time.

We define $\tilde{\Omega}$ as the set of all $(x, v) \in \Omega$ such that the trajectory starting from (x, v) does not contain any grazing collision, in the past or the future. Notice that $\tilde{\Omega}$ is a residual set of full measure, stable under the flow ϕ_t , and that ϕ_t is C^∞ on $\tilde{\Omega}$.

In the special case where D has no boundary, the billiard flow is simply the geodesic flow and $\tilde{\Omega} = \Omega = T^1D$.

We will say that a billiard has *finite horizon* if every trajectory hits the boundary at least once.

Uniform hyperbolicity. We define uniform hyperbolicity in the case of billiards. This definition is given in a more abstract framework in [CM06], but here we adapt it directly to billiard flows.

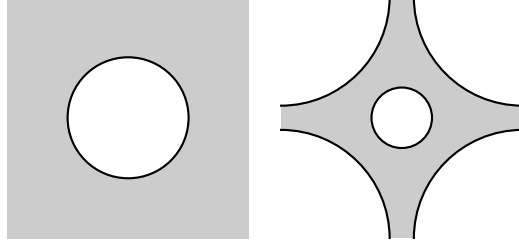


Figure 3: On the left, a dispersing billiard in \mathbb{T}^2 with infinite horizon. On the right, a dispersing billiard in \mathbb{T}^2 with finite horizon.

Definition 1. The billiard flow ϕ_t is *uniformly hyperbolic* if at each point $x \in \tilde{\Omega}$, there exists a decomposition of $T_x\Omega$, stable under the flow,

$$T_x\Omega = E_x^0 \oplus E_x^u \oplus E_x^s$$

where $E_x^0 = \mathbb{R} \frac{d}{dt} \Big|_{t=0} \phi_t(x)$, such that

$$\|D\phi_x^t|_{E_x^s}\| \leq a\lambda^t, \quad \|D\phi_x^{-t}|_{E_x^u}\| \leq a\lambda^t$$

(for some $a > 0$ and $\lambda \in (0, 1)$, which do not depend on x).

Remark. If the billiard D has no wall (which means that the billiard flow is a geodesic flow), we may use the word *Anosov* instead of *uniformly hyperbolic*.

2 Results

In this paper, we give a sufficient condition for a (nonflat) billiard to be uniformly hyperbolic.

2.1 The case of geodesic flows

First, let us consider the case where D has no boundary: the billiard flow is simply the geodesic flow on D . All surfaces with negative curvature have an Anosov geodesic flow: according to Arnold and Avez [AA67], the first proof of this fact goes back to 1898 [Had98]. Later, it was extended to all manifolds with negative sectional curvature (a modern proof is available in [KH95]). But the negative curvature assumption is not necessary for a geodesic flow to be Anosov. To prove that a geodesic flow is Anosov, one may examine the solutions of the Riccati equation

$$u'(t) = -K(t) - u^2(t)$$

where K is the Gaussian curvature of the surface, and use the following criterion:

Theorem 2. *Let M be a closed surface. Assume that there exists $t_0 > 0$ such that for any geodesic $\gamma : [0, 1] \rightarrow M$, and any solution u of the Riccati equation along this geodesic such that $u(0) = 0$, u is well-defined on $[0, t_0]$ and $u(t_0) > 0$. Then the geodesic flow $\phi_t : T^1M \rightarrow T^1M$ is Anosov.*

Theorem 2 was mentioned in [DP03] and [MP13], but as far as we know, no detailed proof was available. In [Kou16a], we apply Theorem 2 to give new examples of surfaces whose geodesic flow is Anosov while their curvature is not negative everywhere.

In fact, it is possible to improve this theorem by considering an increasing sequence of times $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$:

Theorem 3. *Let M be a closed surface. Assume that there exist $m > 0$ and $C > c > 0$ such that for any geodesic $\gamma : \mathbb{R} \rightarrow M$, there exists an increasing sequence of times $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ with $c \leq t_{k+1} - t_k \leq C$, such that the solution u of the Riccati equation with initial condition $u(t_k) = 0$ is defined on the interval $[t_k, t_{k+1}]$, and $u(t_{k+1}) > m$. Then the geodesic flow $\phi_t : T^1M \rightarrow T^1M$ is Anosov.*

Notice that Theorem 2 is immediately deduced from Theorem 3 by choosing a constant step $t_{k+1} - t_k$. Theorem 3 is used in [Kou16b] to obtain a surface of genus 12 embedded in \mathbb{S}^3 with Anosov geodesic flow.

2.2 The case of billiards

Now we consider the general case, in which D may have a boundary.

For billiards, we consider a generalized version of the Riccati equation. We say that u is a solution of this equation if:

1. in the interval between two collisions, $\dot{u}(t) = -K(t) - u(t)^2$;
2. when the particle bounces against the boundary at a time t , u undergoes a discontinuity: we have $u(t^+) = u(t^-) - \frac{2\kappa}{\sin \theta}$, where κ is the geodesic curvature of the boundary of D , and θ is the angle of incidence².

We are now ready to state the main result of this paper:

Theorem 4. *Consider a (not necessarily flat) billiard D . Assume that there exist positive constants A, m, c and C such that for any trajectory γ with $\gamma(0) \in \tilde{\Omega}$, there exists an increasing sequence of times $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfying $c \leq t_{k+1} - t_k \leq C$, such that for any $k \in \mathbb{Z}$, the solution u of the Riccati equation with initial condition $u(t_k^+) = 0$ satisfies $u(t^+) \geq -A$ for all $t \in [t_k, t_{k+1}]$, and $u(t_{k+1}^+) > m$. Also assume that for each $k \in \mathbb{Z}$, there is no collision in the interval $(t_k - c, t_k)$, and at most one collision in the interval $(t_k, t_{k+1}]$. Then the billiard flow on D is uniformly hyperbolic.*

Notice that in the particular case where D has no boundary, Theorem 4 becomes exactly Theorem 3. Thus we only need to prove Theorem 4, which will be done in Section 5.

²The notation $u(t^+)$ stands for $\lim_{h \rightarrow 0, h > 0} u(t + h)$, and likewise $u(t^-) = \lim_{h \rightarrow 0, h < 0} u(t + h)$.

2.3 Applications

We will explain how Theorem 4 can be applied to obtain immediately two famous results: Theorems 5 and 6. Theorem 4 unifies these two theorems, which are both well-known independently. See [Kou16b] for a completely new application of Theorem 4.

Theorem 5. *Let M be a closed Riemannian surface with nonpositive curvature. Assume that every geodesic in M contains a point where the curvature is negative. Then, the geodesic flow on M is Anosov.*

Theorem 5 may also be obtained directly, without using Theorem 2, from Proposition 3.10 of [Ebe73]. Hunt and MacKay [HM03] used this result to exhibit the first Anosov physical system.

For billiards, we will prove the following counterpart of Theorem 5, which is essentially due to Sinai [Sin70]:

Theorem 6. *If D is a smooth dispersing flat billiard in \mathbb{T}^2 with finite horizon, then the billiard flow is uniformly hyperbolic in $\tilde{\Omega}$.*

2.4 Consequences of uniform hyperbolicity

It is shown in [PS72] that (smooth) volume-preserving Anosov flows are ergodic: every invariant subset has either zero or full measure. It was shown later (see [Dol98] and [Kli74]) that Anosov geodesic flows are even exponentially mixing.

As for billiard flows, in the flat case only, Sinai proved ergodicity for smooth dispersing billiards with finite horizon in [Sin70]. It was shown in [BDL15] that such flows are exponentially mixing.

The consequences of uniform hyperbolicity in the nonflat case are still unknown.

2.5 Structure of the paper

In Section 3, we prove a *cone criterion*, following the ideas of Wojtkowski [Woj85]. In Section 4, we study Jacobi fields in (not necessarily flat) billiards. The tools which are introduced in Sections 3 and 4 are used in Section 5 to prove Theorem 4. Finally, the two applications are given in Section 6.

3 The cone criterion

Definition 7. Consider a Euclidean space E .

A *cone*³ in E is a set C such that there exist a decomposition $E = F \oplus G$ and a real number $\alpha \geq 0$ such that

$$C = \{(x, y) \in F \oplus G \mid \|x\| \leq \alpha \|y\|\}.$$

³The word “cone” has several different meanings in mathematics: here we take the same definition as [KH95].

The number $\arctan \alpha$ is called the *angle* of the cone.

Two cones C_1, C_2 are said to be *supplementary* if they correspond to decompositions $E = F_1 \oplus G_1$ and $E = F_2 \oplus G_2$ such that $F_1 = G_2$ and $F_2 = G_1$.

Proposition 8. Consider a sequence of invertible linear mappings $A_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $k \in \mathbb{Z}$, and a sequence of supplementary cones C_k and D_k , corresponding to the decomposition $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. Assume that there exist $a > 0$, $\lambda > 1$ such that for all $k \in \mathbb{Z}$:

1. $A_k(C_k) \subseteq C_{k+1}$ (invariance in the future),
2. $\|A_{k-1} \circ \dots \circ A_{k-i}(v)\| \geq a\lambda^i \|v\|$ for all $i \geq 0$ and $v \in C_{k-i}$ (expansion in the future),
3. $A_k^{-1}(D_{k+1}) \subseteq D_k$ (invariance in the past),
4. $\|A_k^{-1} \circ \dots \circ A_{k+i-1}^{-1}(v)\| \geq a\lambda^i \|v\|$ for all $i \geq 0$ and $v \in D_{k+i}$ (expansion in the past).

Then

$$E_k^u = \bigcap_{i=0}^{+\infty} A_{k-1} \circ \dots \circ A_{k-i}(C_{k-i})$$

is an m -dimensional subspace contained in C_k , and

$$E_k^s = \bigcap_{i=0}^{+\infty} A_k^{-1} \circ \dots \circ A_{k+i-1}^{-1}(D_{k+i})$$

is an $(n - m)$ -dimensional subspace contained in D_k .

Proof. For all $i \geq 0$, $A_{k-1} \circ \dots \circ A_{k-i}(C_{k-i})$ is a cone, which contains a vector space V_i of dimension m . Thus, the intersection E_k^u contains a vector space V of dimension m (for example, consider a converging subsequence of orthonormal bases of V_i). Assume that there exists $w \in E_k^u \setminus V$. Then there exists $v \in V$ and $t \in \mathbb{R}$ such that $v + tw \in \{0\} \times \mathbb{R}^{n-m}$ (notice also that $tw \in E_k^u$). Since $A_{k-i}^{-1} \circ \dots \circ A_{k-1}^{-1}(tw)$ and $A_{k-i}^{-1} \circ \dots \circ A_{k-1}^{-1}(v)$ lie in E_{k-i}^u , Assumption 2 gives us:

$$\|A_{k-i}^{-1} \circ \dots \circ A_{k-1}^{-1}(tw)\| \leq \frac{1}{a\lambda^i} \|tw\| \xrightarrow{k \rightarrow +\infty} 0,$$

$$\|A_{k-i}^{-1} \circ \dots \circ A_{k-1}^{-1}(v)\| \leq \frac{1}{a\lambda^i} \|v\| \xrightarrow{k \rightarrow +\infty} 0,$$

but at the same time, since $v + tw \in D_k$, Assumption 4 gives:

$$\|A_{k-i}^{-1} \circ \dots \circ A_{k-1}^{-1}(v + tw)\| \geq a\lambda^i \|v + tw\| \xrightarrow{k \rightarrow +\infty} +\infty,$$

which contradicts the triangle inequality.

One obtains the result for E_k^s in the same way. □

Theorem 9. Let $A_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ (with $k \in \mathbb{Z}$) be a sequence of 2×2 matrices, with determinant ± 1 . Fix $\epsilon > 0$, and consider the cone C_ϵ of all vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ such that $\epsilon y \leq x \leq \frac{1}{\epsilon} y$. Assume that for all k , and all $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with $xy > 0$,

$$A_k v \in C_\epsilon.$$

Then, there exist $a > 0$ and $\lambda > 1$ such that for all $k \in \mathbb{Z}$, for all $i \geq 0$ and $v \in C_\epsilon$,

$$\|A_{k-1} \circ \dots \circ A_{k-i}(v)\| \geq a \lambda^i \|v\|.$$

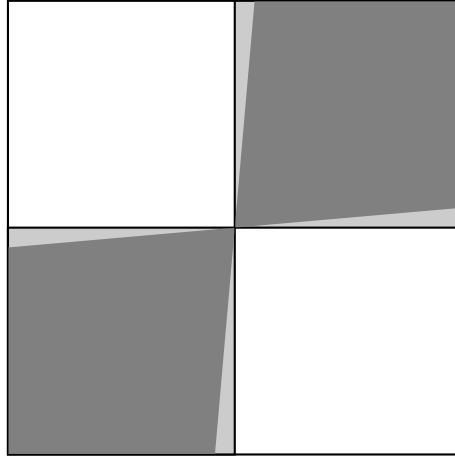


Figure 4: Each A_k maps the cone $xy > 0$ (in grey) into the smaller cone C_ϵ (in dark grey).

Proof. On the basis of Wojtkowski's idea [Woj85], instead of proving expansion directly for the Euclidean norm, we consider the function

$$N : C_\epsilon \rightarrow \mathbb{R}_{\geq 0} \\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \sqrt{xy}.$$

Notice that N is equivalent to the Euclidean norm on C_ϵ , i.e. there exists $M > 0$ such that for all $v \in C_\epsilon$,

$$\frac{1}{M} \|v\| \leq N(v) \leq M \|v\|,$$

because $\frac{\epsilon}{2}(x^2 + y^2) \leq xy \leq \frac{2}{\epsilon}(x^2 + y^2)$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in C_\epsilon$.

We are going to show that for all $k \in \mathbb{Z}$ and $v \in C_\epsilon$, $N(A_k v) \geq \frac{1}{1-\epsilon^2} N(v)$. With the equivalence of norms, this will complete the proof.

Let $k \in \mathbb{Z}$. We may assume that $\det(A_k) = 1$, by multiplying A_k by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the left. Moreover, we may assume that all the coefficients of A_k are positive, by multiplying A_k by $-\text{Id}$.

Notice that the two vectors $A_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_k \\ c_k \end{pmatrix}$ and $A_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_k \\ d_k \end{pmatrix}$ are in the cone C_ϵ , by continuity of A_k .

Then for $v = \begin{pmatrix} x \\ y \end{pmatrix} \in C_\epsilon$:

$$\begin{aligned} N(A_k v) &= (a_k x + b_k y)(c_k x + d_k y) \\ &\geq (a_k d_k - b_k c_k)xy + 2b_k c_k xy \\ &\geq (1 + 2b_k c_k)N(v) \end{aligned}$$

But $a_k d_k - b_k c_k = 1$ and $a_k \leq \frac{1}{\epsilon} b_k, d_k \leq \frac{1}{\epsilon} c_k$, so that $b_k c_k \geq \frac{1}{1-\epsilon^2} - 1$.

Finally, $N(A_k v) \geq \frac{1}{1-\epsilon^2} N(v)$. □

4 Jacobi fields

4.1 Jacobi fields for geodesic flows

In this section, we consider a smooth Riemannian manifold (M, g) . To show that a geodesic flow is hyperbolic, one has to study how the geodesics move away from (or closer to) each other. Thus, one considers small variations of a given geodesic.

Definition 10. Consider a geodesic $\gamma : (a, b) \rightarrow M$. Consider a geodesic variation of γ , i.e. a smooth function

$$f(t, s) : (a, b) \times (c, d) \rightarrow M$$

such that $f(., 0)$ is the geodesic γ , and for all $s \in (c, d)$, $f(., s)$ is a geodesic.

The vector field $Y = \frac{\partial f}{\partial s}$ along the curve $\gamma(t)$ is called an *infinitesimal variation of γ* .

Proposition 11. Any infinitesimal variation of γ is a solution of the Jacobi equation:

$$\ddot{Y} = -R(\dot{\gamma}, Y)\dot{\gamma},$$

where R is the Riemann tensor. The solutions of the Jacobi equation are called Jacobi fields.

Proof. Let ∇ be the Levi-Civita connection of (M, g) . Since $\left[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right] = 0$, one obtains

$$\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} = \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}$$

so that (with $s = 0$):

$$\ddot{Y} = \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} = \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}.$$

On the other hand (still for $s = 0$),

$$\begin{aligned} R(\dot{\gamma}, Y)\dot{\gamma} &= R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial t} \\ &= \nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} - \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} - \nabla_{\left[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right]} \frac{\partial f}{\partial t}. \\ &= -\nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} \end{aligned}$$

Thus, $\ddot{Y} = -R(\dot{\gamma}, Y)\dot{\gamma}$. □

Proposition 12. *Every Jacobi field along a geodesic γ is an infinitesimal variation of γ .*

Proof. Here, we copy the proof of [KN63].

Let $t_1 \in (a, b)$. For any t_2 close enough to t_1 , any solution of the Jacobi equation is determined by its values at t_1 and t_2 (since it is a second-order linear equation).

Let Y be a solution of the Jacobi equation along γ . For $i = 1, 2$, let $h_i(s)$ ($s \in (-\epsilon, \epsilon)$) be a curve such that $(h_i(0), h'_i(0)) = (\gamma(t_i), X(t_i))$. If necessary, take a smaller ϵ , and choose t_2 even closer to t_1 , so that there exists for each s a unique geodesic $f(., s)$ ($t \in (a, b)$), through $h_1(s)$ and $h_2(s)$, of minimal length between $h_1(s)$ and $h_2(s)$. Let X be the Jacobi field $\frac{\partial f}{\partial s}$ along γ . Since X and Y are two solutions of the Jacobi equation which coincide at t_1 and t_2 , they are equal. Thus, Y is an infinitesimal variation of γ . □

We will now be interested in *orthogonal* Jacobi fields:

Lemma 13. *If $Y(t)$ and $\dot{Y}(t)$ are orthogonal to $\dot{\gamma}$ for some $t \in \mathbb{R}$, then they remain orthogonal for all $t \in \mathbb{R}$.*

Proof. We compute

$$\begin{aligned} \frac{\partial}{\partial t} g(\dot{Y}, \dot{\gamma}) &= g(\nabla_{\frac{\partial f}{\partial t}} \dot{Y}, \dot{\gamma}) + g(\dot{Y}, \nabla_{\frac{\partial f}{\partial t}} \dot{\gamma}) \\ &= g(\nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} \dot{\gamma}, \dot{\gamma}) + 0 \\ &= 0. \end{aligned}$$

This concludes the proof of the lemma. □

From now on, assume that M has dimension 2, that γ is a unit speed geodesic, and that Y is a Jacobi field which is orthogonal to $\dot{\gamma}$. Choose an orientation of the normal bundle of γ in M (which has dimension 1), i.e. a vector $e(t) \in T_{\gamma(t)}^1 M$ orthogonal to $\dot{\gamma}(t)$, so that $Y(t)$ is identified by one real coordinate, noted $y(t) = g(Y(t), e(t))$.

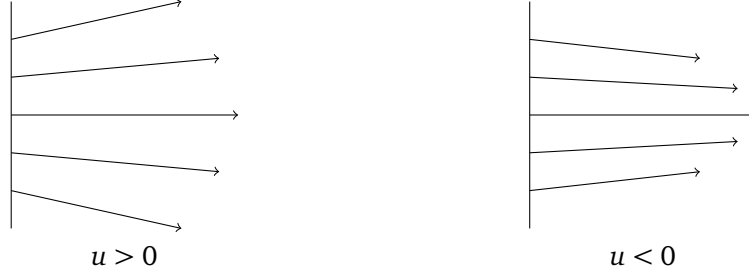
The quantity \dot{y} satisfies

$$\dot{y} = \frac{\partial f}{\partial t} \cdot g(Y, e) = g(\nabla_{\frac{\partial f}{\partial t}} Y, e) + g(Y, \nabla_{\frac{\partial f}{\partial t}} e) = g(\nabla_{\frac{\partial f}{\partial t}} Y, e).$$

Thus:

$$\dot{y} = g(\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s}, e) = g(\nabla_{\frac{\partial f}{\partial s}} \dot{\gamma}, e).$$

In other words, \dot{y} measures the infinitesimal variation of the vector $\dot{\gamma}$ with respect to s . Thus, when y and \dot{y} have the same sign, the Jacobi field is diverging: the geodesics go away from each other. When y and \dot{y} have opposite signs, the Jacobi field is converging. We will consider the ratio $u = \frac{\dot{y}}{y}$, when it is well-defined (i.e. $y \neq 0$), to measure the convergence rate.



Proposition 14. When it is well-defined, u is a solution of the Riccati equation:

$$\dot{u}(t) = -K(\gamma(t)) - u^2(t).$$

where K is the Gaussian curvature.

Proof. In dimension 2, the Riemann tensor may be written

$$\langle R(a, b)c \mid d \rangle = K \cdot (g(a, c)g(d, b) - g(a, d)g(c, b)).$$

Thus, in the case of a unit speed geodesic and of an orthogonal Jacobi field, the vector $R(\dot{\gamma}, Y)\dot{\gamma}$ is orthogonal to $\dot{\gamma}$, and its coordinate is Ky . The Jacobi equation then becomes:

$$\ddot{y} = -Ky.$$

Thus,

$$\dot{u} = \frac{d}{dt} \left(\frac{\dot{y}}{y} \right) = \frac{\ddot{y}y}{y^2} - \frac{(\dot{y})^2}{y^2} = -K - u^2.$$

□

The solutions of this equation are not always defined for all times: it may happen that $u(t)$ explodes to $-\infty$ in positive time (or to $+\infty$ in negative time). This corresponds to the phenomenon of convergence of the wavefront: up to order 1, all the geodesics of the infinitesimal variation “gather at one point”. In most cases, the Jacobi field becomes divergent just after the convergence point (Figure 5).

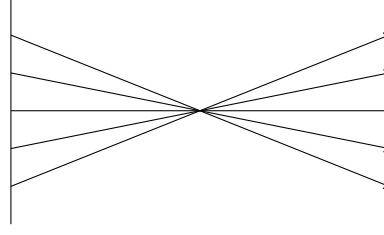


Figure 5: u is not well-defined at the convergence point.

4.2 Jacobi fields for billiards

Recall that a smooth billiard D is a compact subset of a Riemannian manifold (M, g) , such that D has a smooth boundary while M has no boundary.

Consider a billiard trajectory γ and a unit speed variation of this trajectory

$$f(t, s) : (a, b) \times (c, d) \rightarrow D$$

(defined for all times $t \in (a, b)$, except for the collision times) such that $f(., 0)$ is the trajectory γ and for each $s \in (c, d)$, $f(., s)$ is a billiard trajectory.

By analogy with the case of geodesic flows, we shall call⁴ “Jacobi field” the vector field $Y = \frac{\partial f}{\partial s}$ along the curve γ . Inside the billiard, Y satisfies the equation $\ddot{Y}(t) = K(t)Y(t)$, where $K(t)$ is the curvature at the point $\gamma(t)$ (if the billiard is flat, then $\ddot{Y}(t) = 0$). At a (non-grazing) collision time, with an angle of incidence $\theta \in (0, \pi/2]$, Y undergoes a discontinuity, which we are now going to study.

Consider a smooth map $s \mapsto \tau(s)$ such that $\tau(s)$ is a collision time of $f(., s)$ for all $s \in (c, d)$ (reducing the interval (c, d) if necessary). The collision occurs on some component Γ of the boundary ∂D : assume that $r \mapsto \Gamma(r)$ is a parametrization by arc length and define $r(s)$ so that $\Gamma(r(s))$ is the point where the collision occurs for each $s \in (c, d)$. The parametrization of Γ is chosen so that $g(\dot{\gamma}(\tau(0)^-), \frac{d\Gamma}{dr}(r(0))) \geq 0$. As in Section 4.1, choose a section $t \mapsto e_1(t)$ of the unit normal bundle of the trajectory $t \mapsto \gamma(t)$, such that

$$g\left(e_1(\tau(0)^-), \frac{d\Gamma}{dr}(r(0))\right) \leq 0 \quad \text{and} \quad g\left(e_1(\tau(0)^+), \frac{d\Gamma}{dr}(r(0))\right) \geq 0.$$

Define

$$y_{\perp}(t) = g(Y(t), e_1(t)) \quad \text{and} \quad y_{\parallel}(t) = g(Y(t), \dot{\gamma}(t)).$$

Proposition 15. Writing $y_{\perp}^{\pm} = y_{\perp}(\tau(0)^{\pm})$, and defining in the same way y_{\parallel}^{\pm} , we have:

$$y_{\perp}^+ = -y_{\perp}^- \quad \text{and} \quad y_{\parallel}^+ = y_{\parallel}^-.$$

⁴This terminology is commonly used for billiards in the literature: see [Don91] or [Woj94].

Proof. On the one hand,

$$\frac{d}{ds}\Gamma(r(s)) = \frac{y_{\perp}^{-}}{\sin \theta} \quad \text{and} \quad \frac{d}{ds}\Gamma(r(s)) = -\frac{y_{\perp}^{+}}{\sin \theta},$$

so $y_{\perp}^{+} = -y_{\perp}^{-}$.

On the other hand,

$$\frac{d}{ds}\tau(s) = \frac{-y_{\perp}^{-}}{\tan \theta} + y_{\parallel}^{-} \quad \text{and} \quad \frac{d}{ds}\tau(s) = \frac{y_{\perp}^{+}}{\tan \theta} + y_{\parallel}^{+},$$

so $y_{\parallel}^{+} = y_{\parallel}^{-}$. □

From now on, we consider a *perpendicular Jacobi field*, that is, we assume that $y_{\parallel}^{-} = 0$. Proposition 15 implies that $y_{\parallel}^{+} = 0$: in other words, any perpendicular Jacobi field remains perpendicular after a collision. We will write $y(t) = y_{\perp}(t)$ and define $u(t) = \dot{y}(t)/y(t)$.

Proposition 16. *Assume that the geodesic variation f corresponds to an orthogonal Jacobi field.*

At a collision,

$$\begin{aligned} y^{+} &= -y^{-} \\ \dot{y}^{+} &= -\dot{y}^{-} + \frac{2\kappa}{\sin \theta} y^{-} \\ u^{+} &= u^{-} - \frac{2\kappa}{\sin \theta} \end{aligned}$$

where κ is the curvature of the boundary and θ is the angle of incidence.

Proof. The first equality was already proved in Proposition 15. To obtain the next equality, consider the billiard reflection law:

$$\left\langle \frac{\partial f}{\partial t}(\tau(s)^+) - \frac{\partial f}{\partial t}(\tau(s)^-) \mid \frac{\partial \Gamma}{\partial r}(r(s)) \right\rangle = 0.$$

After differentiation with respect to s we obtain:

$$\left\langle \dot{Y}^{+} - \dot{Y}^{-} \mid \frac{\partial \Gamma}{\partial r}(r(s)) \right\rangle + \left\langle \frac{\partial f}{\partial t}(\tau(s)^+) - \frac{\partial f}{\partial t}(\tau(s)^-) \mid \nabla_{\frac{\partial \Gamma}{\partial r}} \frac{\partial \Gamma}{\partial r}(r(s)) \cdot \frac{\partial r}{\partial s} \right\rangle = 0.$$

We may now compute:

$$\begin{aligned} \left\langle \dot{Y}^{+} - \dot{Y}^{-} \mid \frac{\partial \Gamma}{\partial r}(r(s)) \right\rangle &= (\dot{y}^{+} + \dot{y}^{-}) \sin \theta, \\ \left\langle \frac{\partial f}{\partial t}(\tau(s)^+) - \frac{\partial f}{\partial t}(\tau(s)^-) \mid \nabla_{\frac{\partial \Gamma}{\partial r}} \frac{\partial \Gamma}{\partial r}(r(s)) \cdot \frac{\partial r}{\partial s} \right\rangle &= 2 \sin \theta \cdot \kappa \cdot \frac{-y^{-}}{\sin \theta}. \end{aligned}$$

Thus:

$$\dot{y}^{+} = -\dot{y}^{-} + \frac{2\kappa}{\sin \theta} y^{-}$$

and since $u = \dot{y}/y$,

$$u^+ = u^- - \frac{2\kappa}{\sin \theta}.$$

□

In particular, positively curved walls decrease the value of u (and tend to make the Jacobi field converge), just as the positive curvature of a Riemannian manifold. Likewise, negatively curved walls make the quantity u increase, as the negative curvature of a manifold.

5 Proof of Theorem 4

We fix the constants A , c , C and m which appear in the statement of the theorem, and assume that $A \geq 2$.

The readers who are only interested in the proof of Theorem 3 may skip Lemmas 18, 19 and 20.

Lemma 17. *Assume that u and v are two solutions of the Riccati equation on an interval $[t^a, t^b]$ with $c/3 \leq t^b - t^a \leq 2C$, such that $0 \leq u(t^a) - v(t^a) \leq \exp(-4AC)$. Assume that $u(t) \geq -A$ for all $t \in [t^a, t^b]$. Then*

$$u(t^b) - v(t^b) \leq (u(t^a) - v(t^a)) \exp(2A(t^b - t^a)).$$

Proof. Let $t^0 = \min \{t \in [t^a, t^b] \mid u(t) - v(t) \geq 2\}$ (with $t^0 = t^b$ if this set is empty).

Then for $t \in [t^a, t^0]$, we have

$$\dot{u}(t) - \dot{v}(t) = -(u(t) + v(t))(u(t) - v(t)) \leq 2A(u(t) - v(t)).$$

and thus

$$u(t) - v(t) \leq (u(t^a) - v(t^a)) \exp(2A(t - t^a)) \leq 1.$$

Thus $t^0 = t^b$ and the result is proved. □

From now on we will assume that $m \leq \min(\exp(-4AC), 1/4)$ and define

$$\eta = \min(m^3/(K_{\max} + 4), c/3),$$

where K_{\max} is the maximum absolute value of the curvature on D .

Lemma 18. *Assume that u is a solution of the Riccati equation on an interval $[t^b, t^b + \eta]$, during which no collision occurs. If $|u(t^b)| \leq 1/2$, then*

$$u(t^b + \eta) \geq \min(1/2, t^b - m^3).$$

Proof. Consider $t^0 = \max \{t \in [t^b, t^b + \eta] \mid |u(t) - u(t^b)| \geq 1\}$ (or $t^0 = t^b$ if this set is empty). Then

$$|u(t^b + \eta) - u(t^0)| = \left| \int_{t^0}^{t^b + \eta} -K(t) - u(t)^2 dt \right| \leq \eta(K_{\max} + 4) \leq m^3.$$

If $t^0 = t^b$, then $u(t^b + \eta) \geq t^b - m^3$. If $t^0 > t^b$, then $u(t^b + \eta) \geq t^0 - m^3 \geq 1 - m^3 \geq 1/2$. This concludes the proof. \square

Lemma 19. Assume that u and v are two solutions of the Riccati equation on an interval $[t^a, t^b]$ with $c/3 \leq t^b - t^a \leq 2C$, with $u(t^a) = 0$ and $v(t^a + \eta) = 0$. Assume that $u(t) \geq -A$ for all $t \in [t^a, t^b]$. Then $v(t^b) \geq u(t^b) - m^2$.

Proof. If $v(t^a) \geq u(t^a)$ then $v(t^b) \geq u(t^b)$ and there is nothing to prove. Therefore we assume that $u(t^a) \geq v(t^a)$. Lemma 18 implies that $u(t^a + \eta) \geq -m^3$ and Lemma 17 shows that $u(t^b) - v(t^b) \leq m^3 \exp(4AC) \leq m^2$. \square

For each $k \in \mathbb{Z}$, define \tilde{t}_k in the following way:

- If there is a collision in the interval $[t_k - c/3, t_k]$, define $\tilde{t}_k = t_k + \eta$.
- If not, let $\tilde{t}_k = t_k$.

Lemma 20. For all $k \in \mathbb{Z}$, the solution of the Riccati equation with initial condition $u(\tilde{t}_k) = 0$ satisfies $u(\tilde{t}_{k+1}) \geq m/2$.

Proof. If $\tilde{t}_k = t_k$ and $\tilde{t}_{k+1} = t_{k+1}$, then there is nothing to prove.

If $\tilde{t}_{k+1} = t_{k+1} + \eta$, then apply Lemma 18: then the solution of the Riccati equation with initial condition $u(t_k) = 0$ satisfies $u(\tilde{t}_{k+1}) \geq m - m^3$. After that, if $\tilde{t}_k = t_k + \eta$, apply Lemma 19. \square

Lemma 21. For all $k \in \mathbb{Z}$, the solution of the Riccati equation with initial condition $u(\tilde{t}_{k+1}) = 0$ is well-defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_k) \leq -m^2/2$.

Proof. For all $k \in \mathbb{Z}$, Lemma 17 implies that the solution of the Riccati equation along γ with initial condition $u(\tilde{t}_k) = -m^2/2$ satisfies $u(\tilde{t}_{k+1}) \geq 0$.

Now, since the sign of the difference between two solutions of the Riccati equation is constant, the solution of the Riccati equation along γ with initial condition $u(\tilde{t}_{k+1}) = 0$ satisfies $u(\tilde{t}_k) \leq -m^2/2$. This proves the lemma. \square

Lemma 22. Consider $t^0 \in \mathbb{R}$ and a solution u of the Riccati equation along a trajectory γ defined on the interval $[t^0 - \eta, t^0]$. If γ has no collision in the time interval $[t^0 - \eta, t^0]$, then $u(t^0) \leq \alpha$, where

$$\alpha = \sqrt{K_{\max}} \frac{1 + e^{-2\sqrt{K_{\max}}\eta}}{1 - e^{-2\sqrt{K_{\max}}\eta}}.$$

Proof. The Riccati equation gives $\dot{u}(t) \leq K_{\max} - u(t)^2$.

Notice that whenever $u(t) > \sqrt{K_{\max}}$, we have $\dot{u}(t) < 0$. Therefore, the conclusion of the lemma is true if $u(t) \leq \alpha$ for some $t \in [t^0 - \eta, t^0]$.

Now we assume that $u(t) \geq \alpha$ for all $t \in [t^0 - \eta, t^0]$. Thus we may write, for $t \in [t^0 - \eta, t^0]$,

$$\frac{\dot{u}(t)}{K_{\max} - u(t)^2} \geq 1$$

which implies, after integration between $t^0 - \eta$ and t^0 :

$$\frac{u(t^0) - \sqrt{K_{\max}}}{u(t^0) + \sqrt{K_{\max}}} \leq e^{-2\sqrt{K_{\max}}\eta} \frac{u(t^0 - \eta) - \sqrt{K_{\max}}}{u(t^0 - \eta) + \sqrt{K_{\max}}} \leq e^{-2\sqrt{K_{\max}}\eta}.$$

Therefore

$$u(t^0) - \sqrt{K_{\max}} \leq e^{-2\sqrt{K_{\max}}\eta} (u(t^0) + \sqrt{K_{\max}})$$

and thus

$$u(t^0) \leq \alpha.$$

□

For each $(x, v) \in \Omega$, the tangent plane $T_{(x,v)}\Omega$ is the direct sum of a vertical and a horizontal subspace $H_{(x,v)} \oplus V_{(x,v)}$, given by the metric g on M . Each of these two spaces is naturally endowed with a norm, respectively g_H and g_V : one equips Ω with the norm $g_T = g_H + g_V$ (in particular, one decides that H is orthogonal to V).

Denote by $W_{(x,v)} \subseteq T_{(x,v)}\Omega$ the plane orthogonal to the direction of the flow ϕ_t , and let $(w, w') \in W_{(x,v)}$. There exists $Y(t)$ a Jacobi field such that $(Y(0), \dot{Y}(0)) = (w, w')$: then the vectors $\dot{Y}(0)$ and $\dot{\gamma}(0)$ are orthogonal, and $(Y(t), \dot{Y}(t)) = D\phi_t(w, w')$. Lemmas 13 and 15 imply that $Y(t)$ remains orthogonal to $\dot{\gamma}(t)$ for all t . In particular, the family of planes $(W_{(x,v)})$ (where (x, v) varies in $\tilde{\Omega}$) is stable under $D\phi_t$.

Consider an element $(x, v) \in \tilde{\Omega}$, and γ the billiard trajectory such that $(\gamma(0), \dot{\gamma}(0)) = (x, v)$. Choose an orientation of $H_{(\gamma(t), \dot{\gamma}(t))} \cap W_{(\gamma(t), \dot{\gamma}(t))}$, i.e. a continuous unit vector $e_1(t)$ in $H_{(\gamma(t), \dot{\gamma}(t))} \cap W_{(\gamma(t), \dot{\gamma}(t))}$. It induces naturally an orientation of $V_{(\gamma(t), \dot{\gamma}(t))}$, given by a continuous unit vector $e_2(t)$ in $V_{(\gamma(t), \dot{\gamma}(t))}$. This orthogonal basis of $W_{(\gamma(t), \dot{\gamma}(t))}$ allows us to identify it to the Euclidean \mathbb{R}^2 .

For $k \in \mathbb{Z}$, set

$$A_k = D_{(\gamma(\tilde{t}_k), \dot{\gamma}(\tilde{t}_k))} \phi_{\tilde{t}_{k+1} - \tilde{t}_k} : W_{(\gamma(\tilde{t}_k), \dot{\gamma}(\tilde{t}_k))} \rightarrow W_{(\gamma(\tilde{t}_{k+1}), \dot{\gamma}(\tilde{t}_{k+1}))}.$$

The A_k are linear mappings with determinant ± 1 , because the flow ϕ_t preserves the Liouville measure.

Lemma 23. *For each $\epsilon > 0$, consider the cones*

$$C_\epsilon^\pm = \left\{ (x, y) \in \mathbb{R}^2 \mid \epsilon y \leq \pm x \leq \frac{1}{\epsilon} y \right\} \quad \text{and} \quad C_0^\pm = \left\{ (x, y) \in \mathbb{R}^2 \mid \pm xy > 0 \right\}.$$

There exists $\epsilon > 0$ such that for all $k \in \mathbb{Z}$,

$$A_k C_0^+ \subseteq C_\epsilon^+ \quad \text{and} \quad A_k^{-1} C_0^- \subseteq C_\epsilon^-.$$

Proof. First, we prove $A_k C_0^+ \subseteq C_\epsilon^+$.

Since the difference between two solutions of the Riccati equation does not change sign, we only need to see that:

1. The solution of the Riccati equation along γ with initial condition $u(\tilde{t}_k) = 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_{k+1}) \geq \epsilon$. By Lemma 20, it is the case for $\epsilon \leq m/2$.
2. Any solution of the Riccati equation along γ with $u(\tilde{t}_k) \geq 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_{k+1}) \leq 1/\epsilon$. It is the case for $\epsilon \leq 1/\alpha$, where α is defined in Lemma 22.

Now, let us prove $A_k^{-1} C_0^- \subseteq C_\epsilon^-$. We need to see that:

1. The solution of the Riccati equation along γ with initial condition $u(\tilde{t}_{k+1}) = 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_k) \leq -\epsilon$. By Lemma 21, it is the case for $\epsilon \leq m^2/2$.
2. Any solution of the Riccati equation along γ with $u(\tilde{t}_{k+1}) \leq 0$ is defined on $[\tilde{t}_k, \tilde{t}_{k+1}]$ and satisfies $u(\tilde{t}_k) \leq 1/\epsilon$. It is the case for $\epsilon \leq 1/\alpha$, where α is defined in Lemma 22 (recall that there is no collision in the interval $[\tilde{t}_k, \tilde{t}_k + \eta]$, according to the assumptions of Theorem 4).

□

Thus the sequences (A_k) and (A_k^{-1}) satisfy the assumptions of Theorem 9, which provides us with two families of cones: one of them satisfies invariance and expansion in the future, while the other satisfies invariance and expansion in the past. Proposition 8 provides distributions E^s and E^u on $\tilde{\Omega}$ which are stable under the flow ϕ_t , and satisfy

$$\forall k \in \mathbb{Z}, \quad \|D_{(x,v)} \phi_{t_k}|_{E^s}\| \leq a\lambda^k \quad \text{and} \quad \|D_{(x,v)} \phi_{t_{-k}}|_{E^u}\| \leq a\lambda^k$$

for some $a > 0$ and $\lambda \in (0, 1)$.

To go from this discrete statement to a continuous statement, notice the following:

Lemma 24. *Consider the set S of all $(t, (x, v)) \in [0, 2C] \times T^1M$ such that the geodesic of length t starting from (x, v) is contained in the billiard D .*

$$\sup_{(t, (x, v)) \in S} \|D\phi_t(x, v)\| < +\infty.$$

Proof. The set S is compact. □

Therefore, increasing a and λ if necessary, we have:

$$\forall t \in \mathbb{R}, \quad \|D_{(x,v)} \phi_t|_{E^s}\| \leq a\lambda^t \quad \text{and} \quad \|D_{(x,v)} \phi_{-t}|_{E^u}\| \leq a\lambda^t.$$

Hence the billiard flow is uniformly hyperbolic, and Theorem 3 is proved.

6 Applications

6.1 Closed surfaces of negative curvature: proof of Theorem 5

In this proof, we will use the lemma:

Lemma 25. *Under the assumptions of Theorem 5, there exist $m > 0$ and $t_0 > 0$ such that every unit speed geodesic $\gamma : [0, t_0] \rightarrow M$ satisfies:*

$$\int_0^{t_0} K(\gamma(t)) \leq -m.$$

Proof. If the conclusion is false, consider a sequence (γ_n) of unit speed geodesics defined on $[-n, n]$, such that for all n ,

$$\int_{-n}^n K(\gamma(t)) \geq -\frac{1}{n}.$$

By the Arzelà-Ascoli theorem and a diagonal argument, one may extract a subsequence of γ_n which converges uniformly on each $[-n, n]$ to a geodesic defined on \mathbb{R} . By dominated convergence, it satisfies $\int_{\mathbb{R}} K(\gamma(t)) = 0$, which contradicts the assumption. \square

Now, consider the values of m and t_0 given by lemma 25, and choose a geodesic γ . We may assume that $m < 1$ and, by dividing the metric of M by a constant if necessary, that $t_0 < 1$. Denote by u the solution of the Riccati equation $u'(t) = -K(t) - u^2(t)$ with $u(0) = 0$: if this solution is defined on $[0, 1]$, let $t_1 = 1$; if not, write $[0, t_1)$ the maximal interval on which the solution is defined. In particular, $t_1 \leq 1$.

Set $t_2 = \sup \{t \in [0, t_1] \mid u(t) \geq m\}$ (with $t_2 = 0$ if this set is empty). Thus, for all $t \geq t_2$,

$$u'(t) = -K(t) - u^2(t) \geq -m^2.$$

If $t_2 = 0$, then for all $t \in [0, t_1)$, using the estimate given by Lemma 25,

$$u(t) = u(0) + \int_0^t u'(x) dx = \int_0^t -K(x) - u^2(x) dx = - \int_0^t K(x) - \int_0^t u^2(x) dx \geq m - m^2.$$

If $t_2 \neq 0$, then for all $t \in [t_2, t_1[$, using the fact that $K(t) \geq 0$,

$$u(t) = u(t_2) + \int_{t_2}^t u'(x) dx \geq u(t_2) + \int_{t_2}^t -u^2(x) dx \geq m - m^2.$$

In both cases, one gets $u(t) \geq m - m^2$ for all $t \in [t_2, t_1[$. Thus, the solution do not blow up to $-\infty$, so $t_1 = 1$, and $u(1) \geq m - m^2$. One may apply Theorem 2: the geodesic flow on M is Anosov and Theorem 5 is proved.

6.2 Sinai billiards: proof of Theorem 6

Lemma 26. *Let D be a flat billiard in \mathbb{T}^2 with finite horizon. Then, there exists t_0 such that every billiard trajectory in $\tilde{\Omega}$ (with unit speed) experiences at least one collision between $t = 0$ and $t = t_0$.*

Proof. Assume that the conclusion is false. Then for all $n > 0$, there exists a billiard trajectory $\gamma_n : \mathbb{R} \rightarrow \mathbb{T}^2$, without collision on $[-n, n]$: we will write $(x_n, v_n) = (\gamma_n(0), \gamma'_n(0))$. Up to extraction, we may assume that (x_n, v_n) has a limit $(x, v) \in \Omega$. The geodesic of \mathbb{T}^2 starting at (x, v) is contained in D , so it is periodic (since it cannot be dense in \mathbb{T}^2) with period T . If it does not intersect the boundary ∂D , then this geodesic is a billiard trajectory without collision, so the billiard does not have finite horizon. Thus, we assume that this geodesic intersects ∂D , and since ∂D is smooth, there is an open ball B_1 which is tangent to the billiard trajectory, such that $B_1 \cap D = \emptyset$. Furthermore, there is another ball B_2 tangent to the geodesic on the other side, such that $B_2 \cap D = \emptyset$ (otherwise, there is an $x' \in D$ close to x such that the trajectory starting at (x', v) has no collision). If $v_n = v$ for some $n \geq T$, then the trajectory starting at (x_n, v_n) (which has period T) has no collision, which again contradicts the finite horizon assumption: thus $v_n \neq v$ for all $n \geq T$. But since (x_n, v_n) tends to (x, v) , this implies that there exists $n \geq 2T$ such that the trajectory starting at (x_n, v_n) intersects B_1 or B_2 in the time interval $[-2T, 2T]$, which contradicts the assumption. \square

Lemma 27. *If D is a flat billiard with finite horizon whose walls have negative curvature, then it satisfies the assumptions of Theorem 4, where the times t_k are the times of collisions.*

Proof. We consider the solution u of the generalized Riccati equation, such that $u(t_k^+) = 0$. On the interval $]t_k, t_{k+1}[$, u is a solution of the equation $u'(t) = -u^2(t)$, so u is equal to 0. Since the walls have positive curvature, $u(t_{k+1}^+) \geq -2\kappa_{\max} > 0$, where κ_{\max} is the maximum curvature of the boundary. \square

Thus, Theorem 4 applies and concludes the proof.

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